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CORRELATION FUNCTION OF THE STRESS FIELD IN AN ELASTIC MEDIUM WITH POINT DEFECTS*

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A two-point correlation function is constructed for the field of elastic stresses in a medium containing a random set of isolated inhomogeneities in the form of inclusions with other elastic properties or cracks. The main scheme of the construction of the correlation functions of elastic fields in a medium with isolated inhomogeneous is elucidated in /1/. Here this scheme is developed for the case of an elastic medium containing a set of point defects by which finite inclusions are modelled in the interest of simplification. The case of a set of point defects on one line is investigated in detail. Results are presented for a numerical solution of this problem for a specific statistical model of a one-dimensional random set of defects.

Two-point correlation functions of elastic fields contain valuable information about the microstress and strain distribution in stochastically inhomogeneous materials. This information is needed to describe important structurally-sensitive processes that occur during strain, for instance, fracture and transfer into the plastic state. The problem of constructing the correlation functions mentioned hardly allows of exact solution in the general case. However, visible results are obtained successfully in a number of important special cases by using different simplifying assumptions. In the case of small fluctuations of the elastic moduli of a medium, the statistical second moments of the elastic fields can be found by confining oneself to the first terms of the perturbation-theory series for the desired functions (the Born approximation) /2,3/. If the properties of the medium are described by Gaussian random fields, then certain infinite subsequences of terms of the series mentioned are summed successfully by using the methods described in /4/, for example. Here, the effective-field method, which, in principle, enables strong elastic moduli fluctuations and significant concentrations of inclusions to be considered, is used to construct the two-point correlation functions.

1. A homogeneous elastic medium with point defects. Suppose a set of isolated inhomogeneities of the inclusion or crack type are contained in an infinite homogeneous elastic medium with moduli c_0. The solution of the problem of the elastic equilibrium of such a medium in an external stress field σ_0 is examined in an approximation for which the inhomogeneities are modelled by point defects, /5/. Each i-th inhomogeneity is here replaced by a dislocation moment with singular density $m^i(x)$ concentrated at the centre of gravity of the inhomogeneities ξ_i (here and below, the tensor subscripts are denoted by Greek letters):

$$m^i_{\alpha\beta}(x) = P^i_{0\alpha\beta\lambda\mu}\bar{\sigma}_i^{\lambda\mu}\delta(x-\xi_i), \quad i=1,2,3,\ldots$$

($\delta(x)$ is the three-dimensional delta function, $x(x_1, x_2, x_3)$ is a point of the medium, and $\bar{\sigma}_i$ is the stress field in which the i-th defect is located. The constant tensor P_0^i depends on the shape, size, and elastic properties of the i-th inclusion. The problem of a single inclusion in a homogeneous external stress field must be solved in order to construct this tensor. The form of the tensor P_0^i is presented in /5/ in the case of ellipsoidal inclusions. The expressions for the tensors P_0^i are henceforth assumed to be known.

Let X denote the discrete set of points ξ_i at which the point defects are located, and let $X(x)$ be a generalized function concentrated in this set

$$X(x) = \sum_i \delta(x-\xi_i) \tag{1.1}$$

Then the stess $\sigma(x)$ and strain $\varepsilon(x)$ fields in a medium with defects can be represented in the following form /5/ ($\varepsilon_0 = c_0^{-1}\sigma_0$):

$$\sigma^{\alpha\beta}(x) = \sigma_0^{\alpha\beta}(x) + \int S^{\alpha\beta\lambda\mu}(x-x') P_{0\lambda\mu\nu\rho}(x') \bar{\sigma}^{\nu\rho}(x') X(x')\,dx' \tag{1.2}$$

$$\varepsilon_{\alpha\beta}(x) = \varepsilon_{0\alpha\beta}(x) + \int K_{\alpha\beta\lambda\mu}(x-x') c_0^{\lambda\mu\nu\rho} P_{0\nu\rho\tau\delta}(x') \bar{\sigma}^{\tau\delta}(x') X(x')\,dx' \tag{1.3}$$

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Here $P_0(x)$ and $\bar{\sigma}(x)$ are smooth functions which take the values P_0^i and $\bar{\sigma}_i$ at the points $x = \xi_i$, $i = 1, 2, 3\ldots$ The kernels $S(x)$ and $K(x)$ of the integral operators S and K in these relationships are expressed in terms of the second derivatives of Green's function for a homogeneous medium c_0. The properties of the operators S and K required below are presented in /1,5/.

The function $\bar{\sigma}(x)$, given on the set X and governing the local fields in which the individual defects are located, satisfies the equation /5/

$$\bar{\sigma}(x) = \sigma_0(x) + \int S(x-x') P_0(x') \bar{\sigma}(x') X(x; x')\, dx', \quad x \in X \tag{1.4}$$

The function $X(x_0; x)$ is defined for $x_0 \in X$ by the equality

$$X(x_0; x) = \sum_{i \neq k} \delta(x - \xi_i), \quad \text{if} \quad x_0 = \xi_k \tag{1.5}$$

If the solution $\bar{\sigma}(x)$ of (1.4) is known, then the stress and strain in a medium with point defects are defined uniquely by relations (1.2) and (1.3). Therefore, $\bar{\sigma}(x)$ is the main unknown of the problem of the interaction of point defects in an elastic medium.

2. A random set of point defects. Now let the point defects form a random set homogeneous in space. We shall later assume the external stress field σ_0 applied to the medium to be constant. Here $\bar{\sigma}(x)$, $\sigma(x)$, and $\varepsilon(x)$ will be homogeneous random fields. We consider the problem of constructing the statistical moments of these random fields and we start with a function $\bar{\sigma}(x)$ given in a discrete set X. The correlation functions of the fields $\sigma(x)$ and $\varepsilon(x)$ can be expressed in terms of the main statistical moments of the functions $\bar{\sigma}(x)$ (see Sect.5).

We introduce the following notation

$$\langle \bar{\sigma}^{\alpha\beta}(x) \mid x\rangle = \bar{\sigma}^{1\alpha\beta}, \quad \langle \bar{\sigma}^{\alpha\beta}(x_1)\bar{\sigma}^{\lambda\mu}(x_2) \mid x_1; x_2\rangle = \bar{\sigma}^{2\alpha\beta\lambda\mu}(x_1 - x_2)$$

Here $\langle\cdot \mid x\rangle$ is the mean over the ensemble of a random set of point defects under the condition $x \in X$; $\langle\cdot \mid x_1; x_2\rangle$ is this mean under the condition $x_1, x_2 \in X$, $x_1 \neq x_2$. In general, $\langle\cdot \mid x_1, x_2, \ldots, x_n; x_{n+1}, \ldots, x_m\rangle$ denotes the mean under the conditions $x_1, \ldots, x_m \in X$, while the point with the dot separates variables that cannot take identical values. For $x_1 \neq x_2$ only two different defects can evidently be at the points x_1 and x_2. Hence, the function $\bar{\sigma}^2(x)$ characterizes the pairwise interaction in a random set of point defects.

We obtain the expression for the mean $\bar{\sigma}^1$ by taking the average of both sides of (1.4) under the condition $x \in X$

$$\bar{\sigma}^1 = \sigma_0 + \int S(x-x') \langle P_0(x') \bar{\sigma}(x') X(x; x') \mid x\rangle\, dx' \tag{2.1}$$

To calculate the mean $\bar{\sigma}^2(x)$ we multiply both sides of (1.4) by $\bar{\sigma}(x_2)$ and we average the result under the conditions $x = x_1$, $x_2 \in X$

$$\bar{\sigma}^2(x_1 - x_2) = \sigma_0 \Phi(x_1 - x_2) + \int S(x_1 - x') \langle P_0(x') \bar{\sigma}(x') \bar{\sigma}(x_2) X(x_1; x') \mid x_1; x_2\rangle\, dx' \tag{2.2}$$

Here, by virtue of (1.4) the mean

$$\Phi(x_1 - x_2) = \langle \bar{\sigma}(x_2) \mid x_2; x_1\rangle \tag{2.3}$$

is represented in the form

$$\Phi(x_1 - x_2) = \sigma_0 + \int S(x_2 - x') \langle P_0(x') \bar{\sigma}(x') X(x_2; x') \mid x_1; x_2\rangle\, dx' \tag{2.4}$$

In addition to the two-point moment $\bar{\sigma}^2(x_1 - x_2)$ we introduce the mean of the tensor product of the field $\bar{\sigma}$ by itself at the point $x_1 \in X$ assuming that there was a defect at the point x_2 $(x_1 \neq x_2)$

$$D(x_1 - x_2) = \langle \bar{\sigma}(x_1) \bar{\sigma}(x_1) \mid x_1; x_2\rangle \tag{2.5}$$

The limits of the functions $D(x)$ and $\Phi(x)$ are denoted, respectively, by D_∞ and Φ_∞ as $|x| \to \infty$. Since the dependence on x_2 in (2.5) and on x_1 in (2.3) vanishes for large $|x_1 - x_2|$, the following equations hold:

$$\Phi_\infty = \langle \bar{\sigma}(x) \mid x\rangle = \bar{\sigma}^1, \quad D_\infty = \langle \bar{\sigma}(x) \bar{\sigma}(x) \mid x\rangle \tag{2.6}$$

We obtain the expression for the function $D(x)$ like the preceding one by multiplying both sides of (1.4) by $\bar{\sigma}(x)$ and taking the average of the result under the appropriate conditions

$$D(x - x_1) = \sigma_0 \Phi(x - x_1) + \int S(x - x') \langle P_0(x') \bar{\sigma}(x') \bar{\sigma}(x) X(x; x') \mid x; x_1\rangle\, dx' \tag{2.7}$$

We now consider the problem of constructing a closed system of equations for the functions $\overline{\sigma}^2(x)$, $D(x)$ and $\Phi(x)$. Here we start from the following hypothesis about the properties of the random field $\overline{\sigma}(x)$: the value of the field $\overline{\sigma}$ at the point $x \in X$ is statistically independent of the properties of the defect at this point (hypothesis H_2). This is the fundamental hypothesis in the effective-field method. Its physical meaning is discussed in /1/.

Using this hypothesis, the mean under the integral sign in (2.1) can be represented in the form

$$\langle P_0(x')\,\overline{\sigma}(x')\,X(x;x')|x\rangle = \langle P_0(x')\,X(x;x')|x\rangle\,\langle\overline{\sigma}(x')|x',\ x\rangle$$

On substituting this relation into (2.1), the quantity $\overline{\sigma}^1$ is expressed in terms of the mean $\langle\overline{\sigma}(x')\mid x', x\rangle$. A closed equation for $\overline{\sigma}^1$ can be obtained by introducing an additional assumption about the structure of the conditional mean

$$\langle\overline{\sigma}(x')\mid x', x\rangle = \langle\overline{\sigma}(x')\mid x'\rangle = \overline{\sigma}^1 \tag{2.8}$$

This approximation is called "quasi-crystalline" /6/.

The solution of the equation obtained here for $\overline{\sigma}^1$ is examined in /5/.

Let $X_{x_1x_2}$ be a set of X from which points incident in x_1 or in x_2 are removed. If

$$X(x_1, x_2; x) = \sum_{\xi_i \in X_{x_1x_2}} \delta(x - \xi_i)$$

then because of the definition (1.5) of the function $X(x_1; x')$, the following equation holds:

$$X(x_1; x') = X(x_1, x_2; x') + \delta(x' - x_2),\quad x_2 \in X \tag{2.9}$$

Let us examine the conditional mean under the integral in (2.2). Taking into account the preceding relationship, we have

$$\begin{aligned}&\langle P_0(x')\,X(x_1;x')\,\overline{\sigma}(x')\,\overline{\sigma}(x_2)\mid x_1; x_2\rangle = \\ &\langle P_0(x')\,\overline{\sigma}(x')\,\overline{\sigma}(x_2)\,X(x_1, x_2; x')\mid x_1; x_2\rangle + \\ &\delta(x' - x_2)\,\langle P_0(x_2)\,\overline{\sigma}(x_2)\,\overline{\sigma}(x_2)\mid x_1; x_2\rangle\end{aligned} \tag{2.10}$$

Using now hypothesis H_2 and an assumption of the type (2.8)

$$\langle\overline{\sigma}(x')\,\overline{\sigma}(x_2)\mid x', x_1; x_2\rangle = \langle\overline{\sigma}(x')\,\overline{\sigma}(x_2)\mid x'; x_2\rangle = \overline{\sigma}^2(x' - x_2) \tag{2.11}$$

the expresions for each of the means on the right side of (2.10) can be represented in the form

$$\langle P_0(x')\,X(x_1, x_2; x')\,\overline{\sigma}(x')\,\overline{\sigma}(x_2)\mid x_1; x_2\rangle = P\,\langle X(x_1, x_2; x')\mid x_1; x_2\rangle\,\overline{\sigma}^2(x' - x_2) \tag{2.12}$$

$$\langle P_0(x_2)\,\overline{\sigma}(x_2)\,\overline{\sigma}(x_2)\mid x_1; x_2\rangle = PD(x_2 - x_1),\quad P = \langle P_0(x)\mid x\rangle \tag{2.13}$$

It is assumed here that the random functions $P_0(x)$ and $X(x)$ are statistically independent.

Substituting the preceding expressions into (2.10), and the result into (2.2), we obtain the final expression for $\overline{\sigma}^2(x)$:

$$\overline{\sigma}^2(x_1 - x_2) = \sigma_0\Phi(x_1 - x_2) + S(x_1 - x_2)\,PD(x_2 - x_1) + \int S(x_1 - x')\,P\overline{\sigma}^2(x' - x_2)\,F(x', x_1, x_2)\,dx' \tag{2.14}$$

$$F(x', x_1, x_2) = \langle X(x_1, x_2; x')\mid x_1; x_2\rangle \tag{2.15}$$

Using hypothesis H_2 and an assumption analogous to (2.8) and (2.11), the mean under the integral in (2.4) can be represented in the form

$$\langle P_0(x')\,\overline{\sigma}(x')\,X(x_1; x')\mid x_1; x_2\rangle = P\Phi(x' - x_2)\,F(x', x_1, x_2) + \delta(x' - x_2)\,P\Phi(x_2 - x_1).$$

Substituting this result into (2.4), we have

$$\Phi(x_1 - x_2) = \sigma_0 + S(x_1 - x_2)\,P\Phi(x_2 - x_1) + \int S(x_1 - x')\,P\Phi(x' - x_2)\,F(x', x_1, x_2)\,dx' \tag{2.16}$$

Transforming the right side of (2.7) in an analogous manner, we can obtain

$$D(x_1 - x_2) = \sigma_0\Phi(x_1 - x_2) + S(x_1 - x_2)\,P\overline{\sigma}^2(x_1 - x_2) + \int S(x_1 - x')\,P\overline{\sigma}^2(x' - x_1)\,F(x', x_1, x_2)\,dx' \tag{2.17}$$

Equations (2.14), (2.16) and (2.17) form a closed system in the three desired functions $\overline{\sigma}^2(x)$, $\Phi(x)$ and $D(x)$. The specific structure of the random set X occurs in these equations in terms of the function $F(x', x_1, x_2)$ defined by relation (2.15). We consider the mean $F(x', x_1, x_2)$ in greater detail.

3. Certain conditional means of the random functions $X(x)$ and $X(x_1;x)$. We initially consider means of the form $\langle X(x_1;x) \mid x_1\rangle$ and $\langle X(x_1;x) \mid x_1;x_2\rangle$. By definition of the conditional means, we have /7/

$$\langle X(x_1;x)\mid x_1\rangle = \frac{\langle X(x_1;x)X(x_1)\rangle}{\langle X(x_1)\rangle} \tag{3.1}$$

$$\langle X(x_1;x)\mid x_1;x_2\rangle = \frac{\langle X(x_1;x)X(x_1;x_2)X(x_1)\rangle}{\langle X(x_1;x_2)X(x_1)\rangle} \tag{3.2}$$

The set X is later assumed to be ergodic. The standard method that will be utilized to construct means of the types (3.1) and (3.2) is to apply the ergodic property and then take the average over the ensemble if necessary. For instance, starting from definition (1.1) of the function $X(x)$, we have

$$\langle X(x)\rangle = \lim_{v\to\infty}\frac{1}{v}\int_V \sum_{\xi_i\in V}\delta(x-\xi_i)\,dx = \frac{1}{v_0} \tag{3.3}$$

Here V is a domain in $\mathbf{R}^3$ that occupies all space in the limit, v is its volume, and v_0 is the mean volume per element of the set X (one defect).

We evaluate the two-point moment of the function $X(x)$. Using the ergodicity property, we obtain

$$\langle X(x)X(x+x_1)\rangle = \lim_{v\to\infty}\frac{1}{v}\int_V \sum_{\xi_i,\xi_j\in V}\delta(x+x_1-\xi_i)\,\delta(x-\xi_j)\,dx = \lim_{v\to\infty}\frac{1}{v}\sum_{\xi_i,\xi_j\in V}\delta(x_1-\xi_i+\xi_j) \tag{3.4}$$

We introduce the random vector $\xi_{ij}=\xi_i-\xi_j$, and let its distribution density be $g_{ij}(x)$. We take the average of relation (3.4) once again over the ensemble of samples X. The above mentioned mean of the separate components in the last sum has the form

$$\langle\delta(x_1-\xi_{ij})\rangle = \int\delta(x_1-\xi)\,g_{ij}(\xi)\,d\xi = g_{ij}(x_1), \quad i\neq j$$

$$\langle\delta(x_1-\xi_{ij})\rangle = \delta(x_1), \quad i\equiv j$$

It is here taken into account that $\xi_{ii}\equiv 0$.

Extracting the components with $i=j$ in (3.4), we obtain

$$\langle X(x)X(x+x_1)\rangle = \frac{1}{v_0}\delta(x_1) + \lim_{v\to\infty}\frac{1}{v}\sum_{\substack{i,j\\(i\neq j)}}^{N} g_{ij}(x_1) \tag{3.5}$$

where N is the number of points incident in the domain V.

Using this result and the obvious equality

$$X(x) = X(x_1;x) + \delta(x-x_1), \quad x_1\in X$$

we find the expression for the mean in the numerator (3.1)

$$\langle X(x_1;x)X(x_1)\rangle = \lim_{v\to\infty}\frac{1}{v}\sum_{\substack{i,j\\(i\neq j)}}^{N} g_{ij}(x-x_1) \tag{3.6}$$

Let us consider the example of a random point set homogeneous in space. Let its elements ξ_m be random vectors of the form $\xi_m = m+\rho_m+r$, where m is the vector of the m-th node of a regular lattice fixed in space, ρ_m are independent random vectors with zero expectations and the same characteristic function $f(k)$, and r is a random vector distributed uniformly in all space that is identical for all m.

We introduce the function

$$g(x) = \frac{1}{(2\pi)^3}\int f(k)f(-k)\,e^{i(k\cdot x)}\,dk \quad (k\cdot x = k^\alpha x_\alpha)$$

The functions $g_{mn}(x)$ in (3.6) are related to the function $g(x)$ for the point set under consideration by the relationship

$$g_{mn}(x) = g(x-m+n) \tag{3.7}$$

It follows from (3.6) and (3.3) that the expression for the mean (3.1) will here have the form

$$\langle X(x_1;\ x)\mid x_1\rangle=\sum_m{}' g(x-x_1-m)$$

where the prime on the summation sign denotes that the component $m=0$ has been omitted.

We will now construct the mean in the numerator (3.2). Using the ergodic property, we have

$$\langle X(x_1;x_1+x_2)X(x_1;x_1+x_3)X(x_1)\rangle=\lim_{v\to\infty}\frac{1}{v}\int_V\sum_{\xi_i,\xi_j,\xi_k\in V}\delta(x_1-\xi_i)\,\delta(x_1+x_2-\xi_j)\,\delta(x_1+x_3-\xi_k)\,dx_1= \tag{3.8}$$

$$\lim_{v\to\infty}\frac{1}{v}\sum_{\substack{\xi_i,\xi_j,\xi_k\in V\\(j,k\neq i)}}\delta(x_2-\xi_{ji})\,\delta(x_3-\xi_{ki})$$

We take the average of (3.8) once again over the ensemble of samples X. Since ξ_{ji} and ξ_{ki} for $j\neq k$ are independent random vectors, their joint distribution function is $g_{ji}(x)\,g_{ki}(x)$. The means of the separate components of the last sum in (3.8) have the form

$$\langle\delta(x_2-\xi_{ji})\,\delta(x_3-\xi_{ki})\rangle=\begin{cases}g_{ji}(x_2)\,g_{ki}(x_3) & \text{when } j\neq k\\ \delta(x_2-x_3)\,g_{ji}(x_3) & \text{when } j=k\end{cases}$$

Hence, we obtain an expression forthe mean (3.2) from (3.6)—(3.8) in the form

$$\langle X(x_1;x)\mid x_1;x_2\rangle=F(x,x_1,x_2)+\delta(x-x_2)$$

Equation (2.9) is used here. The function F is defined by relation (2.15) and in this case has the form

$$F(x,x_1,x_2)=\Big(\sum_m{}' g(x_1-x_2-m)\Big)^{-1}\sum_m{}' g(x-x_1-m)\sum_{n\neq m}{}' g(x_2-x_1-n) \tag{3.9}$$

where the prime on the summation sign denotes that the component $m=0\ (n=0)$ has been omitted.

4. The one-dimensional case.

We will consider the solution of the system of equations (2.14), (2.16) and (2.17) using the example of the plane problem for a system of point defects located on a line L. Let such defects model a system of rectilinear slits (cracks) of random length $2l$ lying on L. The coordinates of the centres of the slits form a homogeneous random set. We shall consider the exernal stress field to be a uniaxial tension in the direction of the normal n to the line of slits and to have the form $\sigma_0^{\alpha\beta}=\sigma_0 n^\alpha n^\beta$, where σ_0 is a scalar.

We note that the state of any defect is determined uniquely in this case by the normal field component $\bar\sigma$, where it follows from symmetry considerations that $\bar\sigma^{\alpha\beta}(x)\,n_\beta=\bar\sigma(x)\,n^\alpha$, where $\bar\sigma(x)$ is a scalar.

In the case of an isotropic medium the tensors $P_{\alpha\beta\lambda\mu}$ and $n_\alpha S^{\alpha\beta\lambda\mu}(x)\,n_\mu$ take the form (x is the coordinate along L)

$$P_{\alpha\beta\lambda\mu}=\frac{(1-\nu)}{\mu}2\pi b^2 n_\alpha\delta_{\beta\lambda}n_\mu,\qquad n_\alpha S^{\alpha\beta\lambda\mu}(x)\,n_\mu=\frac{\mu}{2\pi(1-\nu)}\frac{1}{x^2}\delta^{\beta\lambda}$$

where μ is the shear modulus, ν is Poisson's ratio of the medium, $\delta_{\alpha\beta}$ is the Kronecker delta, and x^{-2} is a generalized functions whose Fourier transform is $-\pi\mid k\mid$ /8/.

Multiplying (2.14), (2.16) and (2.18) on the left and right by the normal n and taking account of the preceding relationships, we arrive at the system of equations $\left(b^2=\frac{1}{2}\langle l^2\rangle\right)$

$$\bar\sigma^2(x)=\sigma_0\Phi(x)+\frac{b^2}{x^2}D(x)+b^2\int_{-\infty}^{\infty}F(x',x,0)\,\bar\sigma^2(x')\frac{dx'}{(x-x')^2} \tag{4.1}$$

$$D(x)=\sigma_0\Phi(x)+\frac{b^2}{x^2}\bar\sigma^2(x)+b^2\int_{-\infty}^{\infty}F(x',x,0)\,\bar\sigma^2(x'-x)\frac{dx'}{(x-x')^2}$$

$$\Phi(x)=\sigma_0+\frac{b^2}{x^2}\Phi(x)+b^2\int_{-\infty}^{\infty}F(x',x,0)\,\Phi(x')\frac{dx'}{(x-x')^2}$$

in the three scalar functions

$$\bar\sigma^2(x)=\langle\bar\sigma(x)\,\bar\sigma(0)\mid x;0\rangle,\quad D(x)=\langle\bar\sigma(x)\,\bar\sigma(x)\mid x;0\rangle,\quad \Phi(x)=\langle\bar\sigma(x)\mid x;0\rangle \tag{4.2}$$

If the defect density tends to zero, the integral terms vanish in these equations and they describe the interaction of two isolated point inhomogeneities. Here the solution of system (4.1) will have the form

$$\bar\sigma^2(x)=D(x)=[\Phi(x)]^2,\quad \Phi(x)=\sigma_0\frac{x^2}{x^2-b^2} \tag{4.3}$$

The expression for $\Phi(x)$ is the normal component of the stress field in which each of the two identical point defects separated by a distance x are stituated /5/. For $x \leqslant b$ solution (4.3) has no physical meaning.

It is pointed out in /5/ that a point set in which the defects can turn out to be located arbitrarily close to each other has no adequate analog in the case of defects of finite size. Hence, to obtain physically incontrovertible results a constraint must be introduced on the possibility of the defects coming together in a random set X. In the case under consideration, for instance, the centres of the cracks which the point defects simulate should not approach each other a distance less than the sum of half their lengths in order not to merge. This circumstance can be taken into account partially if the following stochastic model of a one-dimensional point set is used.

Let x_k be the coordinate of the k-th defect, and let the differences $x_{k+1} - x_k$ be independent random variables for all k, having the very same normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\tau} \exp\left[-\frac{(x-l_0)^2}{2\tau^2}\right] \tag{4.4}$$

Here l_0 is the mean distance between defects, and τ^2 is the variance. If $\tau \to 0$ we obtain a regular chain of defects. In order to limit the probability of the defects coming together we assume τ to be fairly small. Since the quantity having the distribution (4.4) with unit probability in practice lies in the interval $(l_0 - 3\tau, l_0 + 3\tau)$, we will assume $\varkappa = \tau/l_0 < 1/3$.

By using the method utilized in Sect.3, it can be shown that the function $F(x', x_1, x_2)$ in the integrand of system (4.1) has a form analogous to (3.9) in this case

$$F(x', x_1, x_2) = \Big(\sum_{k=-\infty}^{\infty}{}' f_k(x_1 - x_2)\Big)^{-1} \sum_{k=-\infty}^{\infty}{}' f_k(x' - x_1) \sum_{n=-\infty}^{\infty}{}'' f_n(x_1 - x_2) \tag{4.5}$$

$$f_k(x) = \frac{1}{\sqrt{2\pi|k|}\,\tau} \exp\left[-\frac{(x-kl_0)^2}{2|k|\tau^2}\right] \tag{4.6}$$

(the prime on the summation sign denotes that the component $k = 0$ has been omitted,and the double prime denotes that the components $n = 0$ and $n = k$ have been omitted).

Substituting this expression for F into system (4.1), we seek its solution in the form

$$\bar{\sigma}^2(x) = \bar{\sigma}_1^2(x)\left(\frac{x^2}{x^2-b^2}\right)^2, \quad D(x) = D_1(x)\left(\frac{x^2}{x^2-b^2}\right)^2$$
$$\Phi(x) = \Phi_1(x)\frac{x^2}{x^2-b^2} \tag{4.7}$$

This form of the structure of the solution is dictated by the following considerations. The functions $\bar{\sigma}^2$, D and Φ characterize pairwise interaction in a random set of point defects. To a first approximation it can be considered that their form agrees with the corresponding form for two isolated defects (4.3), while the presence of the surrounding point inhomogeneities is reduced to a change in the external field in which these two defects are situated. Hence, we immediately arrive at representation (4.7).

Numerical computations show that the functions $\bar{\sigma}_1^2(x)$, $D_1(x)$ and $\Phi_1(x)$ are approximated well by constants whose values depend on the parameters $p = 2b/l_0$ and $\varkappa$. We denote these constants, respectively, by $\bar{\sigma}_\infty^2(p, \varkappa)$, $D_\infty(p, \varkappa)$ and $\Phi_\infty(p, \varkappa)$, where it can be shown that $\bar{\sigma}_\infty^2 = (\Phi_\infty)^2$.

Curves of $\Phi_\infty(p, \varkappa)$ and $D_\infty(p, \varkappa) - \Phi_\infty^2(p, \varkappa)$ are shown in Figs.1 and 2. The values $p = 1$, 0.8, 0.6, 0.4 correspond to curves 1–4.

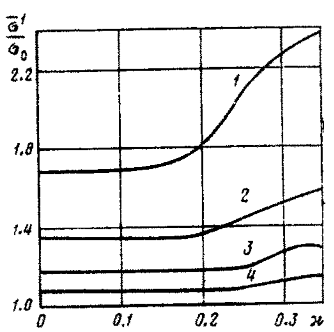

Fig.1

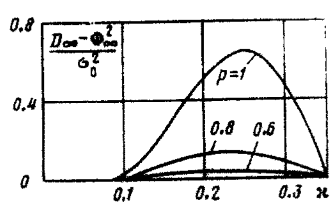

Fig.2

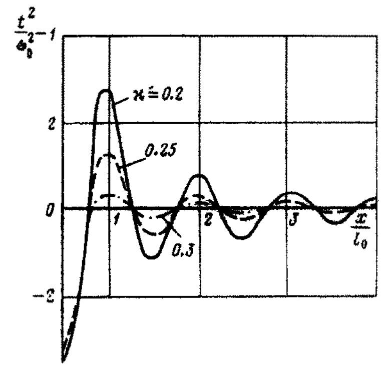

Fig.3

Since $\Phi_\infty = \bar{\sigma}^1$ by virtue of (2.6), the mean value of the field $\bar{\sigma}(x)$ is a maximum for a fairly high relative variance of the distance between the defects $(\varkappa = 0.3 - 0.4)$, as is seen from Fig.1. For $\varkappa = 0.1 - 0.2$ the quantity $\bar{\sigma}^1$ already agrees with the value corresponding to a regular chain of defects $(\varkappa = 0)$.

The difference $D_\infty - (\Phi_\infty)^2$ equals the variance of the field $\bar{\sigma}$. It follows from Fig.2 that the variance is a maximum for $\varkappa = 0.2 - 0.25$ and is practically zero for $\varkappa < 0.1$, which corresponds to a regular structure. For $\varkappa > 1/_3$ the quantity $D_\infty - (\Phi_\infty)^2$ becomes negative. This meaningless physical result is due to the increase in the probability of the defects coming together at distances less than b for large $\varkappa$.

As is seen from (4.7) for $\bar{\sigma}^2(x)$, the correlation radius of the effective field $\bar{\sigma}(x)$ is of the order of the average linear dimension b of the defect and depends slightly on the relative variance of the distance $\varkappa^2$ between defects.

5. Correlation Function of the Elastic Stress Field.

To evaluate the second moment of the stress fields in a medium with point defects, we multiply, as tensors, the expression for $\sigma(x)$ taken at the difference points x_1 and x_2. Taking the average of the result over the ensemble of the set X, we obtain

$$\langle\sigma^{\alpha\beta}(x_1)\sigma^{\lambda\mu}(x_2)\rangle = \sigma_0^{\alpha\beta}\sigma_0^{\lambda\mu} + \int S^{\alpha\beta\nu\rho}(x_1 - x')\langle P_{\sigma\nu\rho\tau\delta}(x')\bar{\sigma}^{\tau\delta}(x')X(x')\rangle dx'\sigma_0^{\lambda\mu} + \tag{5.1}$$
$$\sigma_0^{\alpha\beta}\int S^{\lambda\mu\tau\delta}(x_1 - x')\langle P_{\sigma\tau\delta\nu\rho}(x')\bar{\sigma}^{\nu\rho}(x')X(x')\rangle dx' + \int S^{\alpha\beta\nu\rho}(x_1 - x')dx'\int S^{\lambda\mu\tau\delta}(x_2 - x'')\times$$
$$\langle P_{\sigma\nu\rho\gamma\varphi}(x')P_{\sigma\tau\delta\varepsilon\omega}(x')\bar{\sigma}^{\gamma\varphi}(x')\bar{\sigma}^{\varepsilon\omega}(x'')X(x')X(x'')\rangle dx''$$

Taking hypothesis H_2 into account, we represent the mean under the integrals in this relationship in the form

$$\langle P_0(x')\bar{\sigma}(x')X(x')\rangle = P\bar{\sigma}^1 \tag{5.2}$$
$$\langle P_0(x')P_0(x'')\bar{\sigma}(x')\bar{\sigma}(x'')X(x')X(x'')\rangle =$$
$$\langle X(x''; x')X(x')\rangle P\bar{\sigma}^2(x' - x'')P + \frac{1}{v_0}\delta(x' - x'')\langle P_0^2\rangle D_\infty$$

It is here taken into account that two different defects cannot be at the very same point. Hence, for $x'' = x'$ we have

$$\langle\bar{\sigma}(x')\bar{\sigma}(x'') \mid x', x''\rangle = \langle\bar{\sigma}(x')\bar{\sigma}(x') \mid x'\rangle = D_\infty$$

Substiuting (5.2) into (5.1) and taking into account that the operator S is annihilated by constants /4/, we obtain

$$\langle\sigma^{\alpha\beta}(x)\sigma^{\lambda\mu}(0)\rangle = \sigma_0^{\alpha\beta}\sigma_0^{\lambda\mu} + \frac{1}{v_0}\Pi^{\alpha\beta\lambda\mu}_{\nu\tau\delta\rho}(x)D_\infty^{\nu\tau\delta\rho} + v_0\int\Pi^{\alpha\beta\lambda\mu}_{\nu\tau\delta\rho}(x - x')\bar{\sigma}^{2\nu\tau\delta\rho}(x')\Psi(x')dx' \tag{5.3}$$

$$\Pi^{\alpha\beta\lambda\mu}_{\nu\tau\delta\rho}(x) = \int S^{\alpha\beta\gamma\varphi}(x - x')P_{\gamma\varphi\nu\tau}S^{\lambda\mu\varepsilon\omega}(x')P_{\varepsilon\omega\delta\rho}dx', \quad \Psi(x - x') = \langle X(x; x') \mid x\rangle$$

Therefore, the second statistical moment of the stress field is expressed in terms of conditional moments of the function $\bar{\sigma}(x)$, which is the solution of the system (2.14), (2.16), (2.17). The expression for the second moment of the strain field $\varepsilon(x)$ can be represented in a form analogous to (5.3).

We now consider a one-dimensional set of point defects. We calculate the second moment $t(x)$ of the normal component of the stress tensor when the point x is on the line of defects

$$t(x) = \langle\sigma_{nn}(x)\sigma_{nn}(0)\rangle, \quad \sigma_{nn}(x) = n_\alpha\sigma^{\alpha\beta}(x)n_\beta$$

The expression for the function $t(x)$ in the form

$$t(x) = \sigma_0^2 + \frac{1}{l_0}\pi(x)D_\infty + l_0\int_{-\infty}^{\infty}\pi(x - x')\psi(x')\bar{\sigma}^2(x')dx' \tag{5.4}$$

follows from relation (5.2), where the function $\bar{\sigma}^2(x)$ has the form (4.7) and $\psi(x)$ takes the following form

$$\psi(x) = \sum_{k=-\infty}^{\infty}{}' f_k(x)$$

for the model of a point set considered in Sect.4.

Here $f_k(x)$ is given by (4.6).

The function $\pi(x)$ is the analog of $\Pi(x)$ in (5.3), and has the following form in this case:

$$\pi(x) = b^2 \int_{-\infty}^{\infty} \frac{1}{(x-x')^2} \cdot \frac{1}{(x')^2} dx' = -b^2\delta''(x)$$

Hence, and from (5.4), we finally obtain

$$t(x) = \sigma_0^2 - b^2 \frac{d^2}{dx^2}\left[\frac{1}{l_0} D_\infty \delta(x) + \psi(x)\bar{\sigma}^2(x)\right]$$

A graph of the continuous part of the function $t(x) - \sigma_0^2$ is shown in Fig.3. The presence of a singular component and a singularity at $x = b$ in the correlation function of the random field $\sigma_{nn}(x)$ on the line of defects is due to the replacement of the real cracks by point defects. For a random field of inhomogeneities of finite size the correlation function should be smooth, bounded, and have minimal correlation radius of the order of the mean size of the defect. As a random field of defects approaches a regular lattice, the correlation radius of the stress field grows, as is also seen from Fig.3 (the physically meaningless domain $x \leqslant b$ is not shown in Fig.3).

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NON-AXISYMMETRIC BUCKLING AND POST-CRITICAL BEHAVIOUR OF ELASTIC SPHERICAL SHELLS IN THE CASE OF A DOUBLE CRITICAL VALUE OF THE LOAD*

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The influence of small geometric imperfections of the shape of the middle surface on the non-axisymmetric buckling and initial post-critical behaviour of shallow elastic spherical shells is investigated for a uniform external pressure.

Cases are considered when the least bifurcation load of non-axisymmetric buckling p_0 of the corresponding ideal shell /1/ is a double eigenvalue of the linearized problem, i.e., buckling in two eigen modes occurs. Surfaces of values of the upper critical load as a function of imperfection functionals are constructed by using matrix pivotal condensation /1-7/ and alignment /8-10/ methods for shells with a closed framed edge for $\Lambda = 6.6$ and 9, with a free clamped edge for $\Lambda = 8.045$, and with a fixed hinge-supported edge for $\Lambda = 5.655$ and $\Lambda \to \infty$, where the parameter is $\Lambda = 2[3(1-\nu^2)]^{1/4}(H/h)^{1/2}$, and H is the height of the shell rise, h is its thickness, and ν is Poisson's ratio.

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